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LARGE TIME BEHAVIOUR OF SOLUTIONS OF THE POROUS MEDIA EQUATION WITH ABSORPTION

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ABSTRACT
We study the large time behaviour of nonnegative solutions of the Cauchy problem $u_t = \Delta u^m - u^p$, $u(x,0) = \phi(x)$. Specifically we study the influence of the rate of decay of $\phi(x)$ for large $|x|$, and the competition between the diffusion and the absorption term.

1. Introduction

We consider the Cauchy problem

$$\text{(I)}\quad \begin{cases} u_t = \Delta(u^m) - u^p & \text{in } \mathbf{R}^N \times (0,\infty), \qquad (1.1) \\ u(x,0) = \phi(x) & \text{in } \mathbf{R}^N, \qquad (1.2) \end{cases}$$

in which $m \geqq 1$, $p > 1$, $N \geqq 1$ and ϕ is a given bounded nonnegative function.

The existence and uniqueness of a nonnegative bounded solution u of Problem I — defined in some weak sense — is well established [4, 13].

In this paper we are interested in the behaviour of $u(x,t)$ as $t \to \infty$ and how this is determined by

(i) the competition between the diffusion and the absorption term;

(ii) the asymptotic behaviour of $\phi(x)$ as $|x| \to \infty$.

To specify the asymptotic behaviour of ϕ we introduce a parameter $\alpha > 0$ through the hypothesis

$$\lim_{|x|\to\infty} |x|^\alpha \phi(x) = A$$

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where A is a positive number and the limit is taken in a distributional sense (see section 2).

For $m = 1$, these questions have been discussed by Gmira and Veron [12], Kamin and Peletier [15, 16], Kamin and Ughi [17] and Escobedo and Kavian [9]. For that case the situation is conveniently described by the diagram given in Fig. 1. For the various regions in the p–α plane the large time behaviour of $u(x,t)$ is as follows.

$(p,\alpha) \in I$ [12]:

$$t^{1/(p-1)}u(x,t) \to c^* \qquad \text{as } t \to \infty$$

where $c^* = (1/(p-1))^{1/(p-1)}$, uniformly on sets of the form

$$P_a(t) = \{x \in \mathbf{R}^n : |x| \leqq at^{1/2}\} \qquad a \geqq 0, \quad t \geqq 0.$$

$(p,\alpha) \in III$ [16]:

$$t^{\alpha/2}|u(x,t) - W(x,t)| \to 0 \qquad \text{as } t \to \infty,$$

uniformly on sets P_a. Here W is the solution of the heat equation with initial value $W(x,0) = A|x|^{-\alpha}$.

$(p,\alpha) \in V$ [12]:

$$t^{n/2}|u(x,t) - c_0 E(x,t)| \to 0 \qquad \text{as } t \to \infty$$

uniformly on sets P_a. The function E is the fundamental solution of the heat equation and c_0 is a positive constant which depends on u.

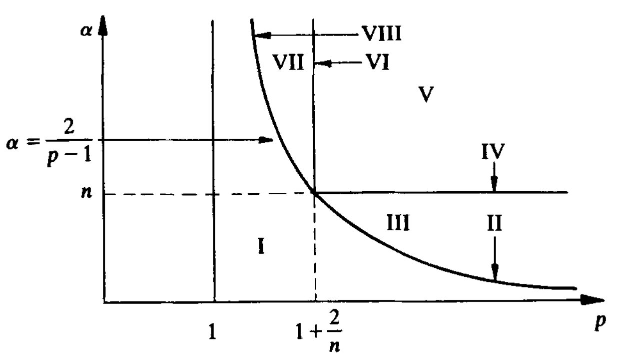

Fig. 1.

The case $(p,\alpha)\in \mathrm{VII}$ has not been completely solved yet. However, if $1<p<1+(2/n)$ and there exist $A>0$, $a>0$ so that

$$\phi(x)\leqq Ae^{-a|x|^2} \qquad \text{in } \mathbf{R}^n,$$

then

$$t^{1/(p-1)}|u(x,t)-V(x,t)|\to 0 \qquad \text{as } t\to\infty,$$

uniformly in $\mathbf{R}^n$. Here V is the — unique — very singular solution of equation (1.1) [6, 9, 15, 18].

For the borderline cases II, IV, VI and VIII we refer to, respectively, [16], [17], [12] and [9].

In this paper we shall extend some of these results to the case $m>1$. It will be necessary to carve up the p–α plane slightly differently (see Fig. 2).

We shall give the asymptotic behaviour of u as $t\to\infty$ for the cases I, III and V. For the case VII the situation is not yet clear even for $m=1$. On the other hand, for case IX the behaviour has already been described in [5]. The borderline cases will not be considered either.

We shall show that

(a) If $(p,\alpha)\in I$, then

$$t^{1/(p-1)}u(x,t)\to c^* \qquad \text{as } t\to\infty$$

uniformly in the sets $\{x\in\mathbf{R}^n : |x|< at^{1/\beta}\}$, where $a\geqq 0$ and

$$\beta=2(p-1)/(p-m).$$

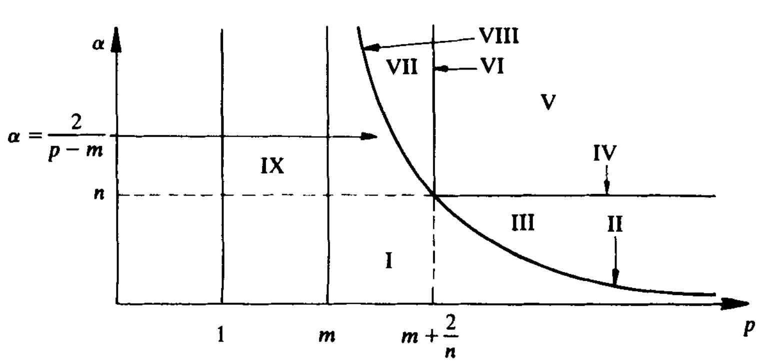

Fig. 2.

(b) If $(p,\alpha)\in III$, then

$$t^{\alpha/\gamma}\,|u(x,t)-W(x,t)|\to 0 \qquad \text{as } t\to\infty$$

uniformly on sets $\{x\in\mathbf{R}^n : |x|<at^{1/\gamma}\}$, where $a\geqq 0$,

$$\gamma=(m-1)\alpha+2$$

and W is the solution of the Porous Media Equation

$$u_t=\Delta(u^m) \tag{1.3}$$

with initial value $W(x,0)=A\,|x|^{-\alpha}$.

(c) If $(p,\alpha)\in V$:

$$t^{1/\delta}\,|u(x,t)-E_{c_0}(x,t)|\to 0 \qquad \text{as } t\to\infty$$

uniformly on sets of the form $\{x\in\mathbf{R}^n : |x|\leqq at^{1/\delta n}\}$. Here

$$\delta=m-1+2/n$$

and E_{c_0} is the Barenblatt–Pattle solution of the Porous Media Equation (1.3) with mass c_0, and

$$c_0=\|\phi\|_{L^1}-\int_0^\infty\int_{\mathbf{R}^n}u^p(x,t)\,dxdt.$$

The results obtained above for solutions of Problem I can readily be generalized to solutions of the equation

$$u_t=\Delta(u^m)-g(u)$$

in which

$$g(0)=0 \quad \text{and} \quad g(s)>0 \qquad \text{for } s>0.$$

This is done in the last section.

Analogous asymptotic estimates for the Porous Media Equation (1.3) have been obtained by Alikakos and Rostamian [1].

2. Preliminaries

We shall write $\mathbf{R}^+=(0,\infty)$, $S=\mathbf{R}^n\times\mathbf{R}^+$ and for $T>0: S_T=\mathbf{R}^n\times(0,T]$ and $S_T^\tau=\mathbf{R}^n\times(\tau,T]$ if $0<\tau<T$. We shall assume that $\phi\geqq 0$ and $\phi\in L^\infty(\mathbf{R}^n)$.

Definition. By a solution of Problem I on $[0,T]$ we shall mean a nonnegative function $u\in L^\infty(S_T)$ which satisfies the identity

$$\int\int_{S_T} [\zeta_t u + \Delta\zeta u^m - \zeta u^p]dxdt + \int_{\mathbf{R}^n} \zeta(x,0)\phi(x)dx = 0 \tag{2.1}$$

for any $\zeta \in C^{2,1}(\bar{S}_T)$ which vanishes for large $|x|$ and at $t = T$.

The existence and uniqueness of such a solution is well established [4, 13]. That it is continuous in S_T was shown in [8].

To characterize the initial values ϕ, we introduce the hypothesis

$$|x|^\alpha \phi(x) \to A \qquad \text{as } |x| \to \infty \tag{H1}$$

in which $\alpha > 0$ and $A > 0$ and the convergence is understood in the following distributional sense: for any $\chi \in C_0^\infty(\mathbf{R}^n)$,

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} \chi(x)|kx|^\alpha \phi(kx)dx = A \int_{\mathbf{R}^n} \chi(x)dx.$$

In the description of the asymptotic form of the solution at large times, we shall encounter two families of special solutions of the Porous Media Equation (1.3).

(1) The Barenblatt–Pattle solutions $E_c(x,t)$:

$$E_c(x,t) = t^{-1/\delta}\left\{\left[a^2 - \frac{(m-1)|x|^2}{2mn\delta t^{2/\delta n}}\right]_+\right\}^{1/(m-1)},$$

where $[z]_+ = \max\{0, z\}$, $\delta = m - 1 + (2/n)$ and a is a constant chosen so that $\int_{\mathbf{R}^n} E_c(x,t)dx = c > 0$ [2].

(2) The solution $W_A(x,t)$ of the problem

$$\text{(II)} \quad \begin{cases} u_t = \Delta(u^m) & \text{in } S, \\ u(x,0) = A|x|^{-\alpha} & \text{in } \mathbf{R}^n \setminus \{0\}, \end{cases}$$

where $A > 0$ and $0 < \alpha < n$. The existence and uniqueness of this solution was established in [3] (see also [7]). By the symmetry properties of Problem II, W_A must be of the form

$$W_A(x,t) = t^{-\alpha/\gamma} f(\eta; A), \qquad \eta = |x|/t^{1/\gamma}, \tag{2.2}$$

where $\gamma = (m-1)\alpha + 2$ and f is the solution of the problem

$$\text{(III)} \quad \begin{cases} (f^m)'' + \dfrac{n-1}{\eta}(f^m)' + \dfrac{1}{\gamma}\eta f' + \dfrac{\alpha}{\gamma} f = 0, & \eta > 0, \\ f \geqq 0 \ (\not\equiv 0), & \eta \geqq 0, \\ f'(0) = 0, \quad \lim_{\eta\to\infty} \eta^\alpha f(\eta, A) = A. & \end{cases}$$

Note that the condition on f at infinity is required by the initial condition. If $x \neq 0$ we have

$$\lim_{t \downarrow 0} W_A(x,t) = \lim_{t \downarrow 0} t^{-\alpha/\gamma} f(\eta, A)$$

$$= |x|^{-\alpha} \lim_{\eta \to \infty} \eta^{\alpha} f(\eta; A)$$

$$= A\,|x|^{-\alpha}.$$

3. Case I

In this section we assume that

$$p > m \geqq 1 \tag{3.1}$$

and that ϕ has the property

$$\lim_{|x| \to \infty} |x|^{2/(p-m)} \phi(x) = \infty. \tag{H2}$$

Thus, if ϕ satisfies (H1), i.e. $\phi(x) \sim A\,|x|^{-\alpha}$ as $|x| \to \infty$, we assume that

$$0 < \alpha < \frac{2}{p-m}. \tag{3.2}$$

THEOREM 1. *Suppose m and p satisfy* (3.1). *Let u be a solution of Problem I in which ϕ satisfies* (H2). *Then*

$$t^{1/(p-1)} u(x,t) \to c^* \qquad \text{as } t \to \infty$$

where $c^* = (1/(p-1))^{1/(p-1)}$ uniformly on sets of the form

$$\{x \in \mathbf{R}^n : |x| \leqq a t^{1/\beta}\}, \qquad a \geqq 0, \quad t \geqq 0$$

where

$$\beta = 2\frac{p-1}{p-m}.$$

To prove this theorem we define the family of functions

$$u_k(x,t) = k^{2/(p-m)} u(kx, k^{\beta} t), \qquad k > 0. \tag{3.3}$$

It is readily seen upon substitution that for every $k > 0$, u_k is a solution of Problem I with initial value

$$\phi_k(x) = k^{2/(p-m)} \phi(kx). \tag{3.4}$$

LEMMA 1. *For every* $k>0$

$$u_k(x,t) \leqq c^* t^{-1/(p-1)} \qquad \text{in } S$$

where c^* *has been defined in Theorem* 1.

Since $c^* t^{-1/(p-1)}$ is a solution of equation (1.1), Lemma 1 follows from the Comparison Principle.

LEMMA 2. *There exists a subsequence* $\{u_{k'}\}$ *and a function* $U \in C(S)$ *such that*

$$u_k(x,t) \to U(x,t) \qquad \text{as } k \to \infty$$

uniformly on compact subsets of S.

PROOF. The uniform upper bound of Lemma 1 implies, by a result of DiBenedetto [8], that the sequence $\{u_k\}$ is equicontinuous on compact subsets of S, enabling us to extract a convergent subsequence.

Lemmas 1 and 2 imply that

$$U(x,t) \leqq c^* t^{-1/(p-1)} \qquad \text{in } S. \tag{3.5}$$

To obtain a lower bound for U we define for any fixed constant $A>0$ the family of truncated initial values

$$\phi_k^A(x) = \min\{\phi_k(x), A\}.$$

We denote the corresponding solutions of Problem I by $v_k^A(x,t)$. Clearly, by the Comparison Principle

$$v_k^A(x,t) \leqq u_k(x,t) \qquad \text{in } \bar{S} \tag{3.6}$$

for every $k>0$ and $A>0$.

Define the function

$$V_A(x,t) = c^* \left(t + \frac{1}{(p-1)A^{p-1}}\right)^{-1/(p-1)};$$

it is the solution of Problem I which corresponds to the uniform initial value $\phi(x) \equiv A$.

LEMMA 3. *Let* $A>0$. *Then*

$$v_k^A(x,t) \to V_A(x,t) \qquad \text{as } k \to \infty$$

for every $(x,t) \in \bar{S}$.

PROOF. Because v_k^A is a solution of Problem I, it satisfies the identity

$$\iint_S \{\zeta_t v_k^A + \Delta\zeta (v_k^A)^m - \zeta (v_k^A)^p\} dxdt + \int_{\mathbf{R}^n} \zeta(x,0)\phi_k^A(x)dx = 0$$

for any $\zeta \in C^{2,1}(\bar{S})$ which vanishes for $|x|$ and t large and for any $k > 0$.

Because $v_k^A \leqq V_A$ in $\bar{S}$, we can pass to the limit through a subsequence:

$$v_{k'}^A(x,t) \to \hat{V}(x,t) \qquad \text{as } k' \to \infty$$

uniformly on compact sets in $\bar{S}$. Moreover, by (H2)

$$\lim_{k\to\infty} \phi_k^A(x) = A$$

for every $x \in \mathbf{R}^n \setminus \{0\}$. Thus by the dominated convergence theorem the identity becomes in the limit

$$\iint_S \{\zeta_t \hat{V} + \Delta\zeta \hat{V}^m - \zeta \hat{V}^p\} dxdt + \int_{\mathbf{R}^n} \zeta(x,0) A dx = 0.$$

Therefore $\hat{V}$ is the — unique — solution V_A of Problem I with initial value $\phi(x) \equiv A$. Thus $\hat{V} = V_A$ and in view of the uniqueness of V_A, the entire sequence $\{v_k^A\}$ must converge to V_A. This completes the proof.

We now let $k \to \infty$ in (3.6). Then, by Lemmas 1, 2 and 3 we obtain

$$V_A(x,t) \leqq U(x,t) \leqq V_\infty(x,t) \qquad \text{in } S.$$

Since the lower bound holds for every $A > 0$ we may conclude that

$$U(x,t) = V_\infty(x,t) \qquad \text{in } S.$$

Thus, by Lemma 1 and the definition of the functions u_k:

$$k^{2/(p-m)} u(kx, k^\beta) \to c^* \qquad \text{as } k \to \infty$$

uniformly on bounded intervals. Therefore, if we set $kx = x'$ and $k^\beta = t'$ we obtain, omitting the primes again,

$$t^{1/(p-1)} u(x,t) \to c^* \qquad \text{as } t \to \infty$$

uniformly on sets $\{x \in \mathbf{R}^n : |x| \leqq at^{1/\beta}\}$, $a \geqq 0$, $t \geqq 0$.

REMARK. For $m = 1$, Theorem 1 was proved in [12]. The proof presented here is somewhat different and seems to be simpler.

4. Case III

In this section we assume that

(4.1) $$p > m + 2/n, \qquad m > 1$$

and that ϕ satisfies (H1) with

(4.2) $$\frac{2}{p-m} < \alpha < n.$$

In addition we impose a uniform bound on ϕ:

(H3) $\exists B > 0$ such that

$$|x|^\alpha |\phi(x)| \leqq B \qquad \text{for all } x \in \mathbf{R}^n.$$

As in the introduction, we write

$$\gamma = (m-1)\alpha + 2.$$

THEOREM 2. *Suppose m and p satisfy* (4.1) *and* ϕ *satisfies* (H1) *and* (H3) *in which* α *satisfies* (4.2). *Let u be the solution of Problem I. Then*

$$t^{\alpha/\gamma}u(x,t) \to f\left(\frac{x}{t^{1/\gamma}}; A\right) \qquad \text{as } t \to \infty$$

uniformly on sets of the form

$$\{x \in \mathbf{R}^n : |x| \leqq at^{1/\gamma}\}, \qquad a \geqq 0.$$

Here f is the nonnegative solution ($f \not\equiv 0$) *of the problem*

$$\text{(III)} \begin{cases} (f^m)'' + \dfrac{n-1}{\eta}(f^m)' + \dfrac{1}{\gamma}\eta f' + \dfrac{\alpha}{\gamma} f = 0, & \eta > 0, \\ f'(0) = 0, & \lim\limits_{\eta \to \infty} \eta^\alpha f(\eta) = A. \end{cases}$$

For $k > 0$ we now define the functions

$$u_k(x,t) = k^\alpha u(kx, k^\gamma t).$$

They are the solutions of the problems

$$(\mathrm{I}_k) \begin{cases} u_t = \Delta(u^m) - k^{-\nu}u^p & (4.3) \\ u(x,0) = \phi_k(x) & (4.4) \end{cases}$$

in which $\nu = \alpha(p-m)-2$ and $\phi_k(x) = k^{\alpha}\phi(kx)$. Note that $\nu > 0$ by assumption (4.2).

As in [16] we can show that there exists an $a > 0$ such that for every $k > 0$

$$u_k(x,t) \leqq W_a\left(x, t + \frac{1}{k^{\gamma}}\right), \tag{4.5}$$

where W_a has been defined in section 2. Recall that $\phi \in L^{\infty}(\mathbf{R}^n)$ is assumed throughout. For the proof of (4.5) (H3) is also needed. Thus, the family of solutions $\{u_k\}$ is uniformly bounded in $\bar{S} \setminus \{(0,0)\}$ whence it is equicontinuous on compact subsets of $\bar{S} \setminus \{(0,0)\}$ [8]. Thus it is possible to find a subsequence $\{u_{k'}\}$ and a function $U \in C(\bar{S} \setminus \{(0,0)\})$ such that

$$u_{k'} \to U \qquad \text{as } k' \to \infty$$

uniformly on compact subsets of $\bar{S} \setminus \{(0,0)\}$.

By passing to the limit in the integral identity (2.1) for the solutions u_k of Problem I_k, we shall show that U is a solution of the problem

$$\text{(II)} \begin{cases} u_t = \Delta(u^m) & \text{in } S, \\ u(x,0) = A\,|x|^{-\alpha} & \text{in } \mathbf{R}^n \setminus \{0\}. \end{cases}$$

However, before we can do that we need some further estimates on the functions u_k.

LEMMA 4. *Suppose ϕ satisfies* (H3). *Then there exist positive constants C, C_1 and C_2, independent of k, such that*

(i)
$$\int_0^{\tau}\int_{B_1} u_k(x,t)\,dx\,dt \leqq C\tau,$$

(ii)
$$\int_0^{\tau}\int_{B_1} u_k^q(x,t)\,dx\,dt \leqq C_1\tau + C_2 \begin{cases} \tau^{\lambda/\gamma} & \text{if } \lambda > 0, \quad \lambda \neq \gamma \\ \tau \log(1/\tau) & \text{if } \lambda = \gamma \\ \log(1 + k^{\gamma}\tau) & \text{if } \lambda = 0 \\ k^{-\lambda} & \text{if } \lambda < 0 \end{cases}$$

where $q > 1$ and $\lambda = n - \alpha q + \gamma$. If $\gamma > 0$ we need the additional assumption $k^2\tau \geqq 1$.

REMARK. As we are interested here in u_k when k is large, the condition $k^2\tau \geqq 1$ poses no restriction.

PROOF. (i) By (4.5),

$$\int_0^\tau \int_{B_1} u_k(x,t)dxdt \leqq \int_0^\tau \int_{B_1} W_a\left(x, t+\frac{1}{k^\gamma}\right)dxdt \tag{4.6}$$
$$\leqq \int_\varepsilon^{\tau+\varepsilon} \int_{B_1} W_a(x,s)dxds$$

where $\varepsilon = 1/k^\gamma$ and $s = t + \varepsilon$. Using the special form (2.2) of W_A, we find that

$$\int_0^\tau \int_{B_1} u_k(x,t)dxdt \leqq |\partial B_1| \int_\varepsilon^{\tau+\varepsilon} s^{(n-\alpha)/\gamma}ds \int_0^{s^{-1/\gamma}} f(\eta)\eta^{n-1}d\eta,$$

where $|\partial B_1|$ denotes the area of ∂B_1. Since f is a solution of Problem III, and hence $\eta^\alpha f(\eta) \to A$ as $\eta \to \infty$,

$$\int_0^{s^{-1/\gamma}} f(\eta)\eta^{n-1}d\eta \leqq K s^{(\alpha-n)/\gamma}, \qquad s > 0$$

for some positive constant K. Thus

$$\int_0^\tau \int_{B_1} u_k(x,t)dxdt \leqq C\tau,$$

where $C = K|\partial B_1|$.

Part (ii) follows in a completely analogous manner.

LEMMA 5. *Let ϕ have the properties* (H1) *and* (H3). *Then U is a solution of Problem II.*

PROOF. By the definition of a solution of Problem I_k we can write for any $\tau \in (0, T)$ and any test function ζ

$$\left(\iint_{S_\tau} + \iint_{S_T^\tau}\right)[\zeta_t u_k + \Delta\zeta u_k^m - k^{-\nu}\zeta u_k^p]dxdt + \int_{\mathbf{R}^n} \zeta(x,0)\phi_k(x)dx = 0$$

where $S_T^\tau = S_T \setminus S_\tau$.

If we now let $k \to \infty$ through the subsequence $\{k'\}$ and use the estimates of Lemma 4 we obtain eventually that

$$\iint_{S_T} (\zeta_t U + \Delta\zeta U^m)dxdt + \int_{\mathbf{R}^n} \zeta(x,0)A|x|^{-\alpha}dx = 0,$$

whence U is a solution of Problem II.

Thus, the sequence $\{u_{k'}\}$ converges to a solution U of Problem II. As we saw in section 2, the unique solution of Problem II is W_A. Therefore $U = W_A$ and, using the uniqueness of W_A again, we may conclude that the entire sequence u_k converges to W_A as $k \to \infty$.

Set $t = 1$. Then

$$u_k(x,1) = k^{\alpha} u(kx, k^{\gamma}) \to W_A(x,1) \qquad \text{as } k \to \infty,$$

uniformly on compact subsets of $\mathbf{R}^n$. Thus, writing $kx = x'$ and $k^{\gamma} = t'$ and dropping the primes again, we obtain

$$\begin{aligned} t^{\alpha/\gamma} u(x,t) &\to W_A(x/t^{1/\gamma}, 1) \\ &= f(\eta; A) \qquad \text{as } t \to \infty \end{aligned}$$

uniformly on sets $\{x \in \mathbf{R}^n : |x| \leqq a t^{1/\gamma}\}$, $a \geqq 0$, $t \geqq 0$.

5. Case V

In this section we assume that

$$p > m + 2/n, \qquad m > 1 \tag{5.1}$$

and that $\phi \in L^1(\mathbf{R})$. Thus, if ϕ satisfies (H1) we assume that $\alpha > n$.

THEOREM 3. *Suppose m and p satisfy* (5.1) *and* $\phi \in L^1(\mathbf{R}^n)$. *Then the solution u of Problem I has the property*

$$t^{1/\delta} |u(x,t) - E_{c_0}(x,t)| \to 0 \qquad \text{as } t \to \infty$$

uniformly on sets of the form $\{x \in \mathbf{R}^n : |x| \leqq a t^{1/\delta n}\}$, $a \geqq 0$. *Here* $\delta = m - 1 + (2/n)$ *and* E_{c_0} *is the Barenblatt–Pattle solution with mass* c_0 *and* c_0 *is given by*

$$c_0 = \|\phi\|_{L^1} - \int_0^{\infty} \int_{\mathbf{R}^n} u^p(x,t)\,dx\,dt.$$

Let $u(x,t)$ be the solution of Problem I. Then, to prove Theorem 3, we now consider the family of functions

$$u_k(x,t) = k^n u(kx, k^{\delta n} t), \qquad k > 0.$$

For every $k > 0$, u_k is a solution of the problem

$$\begin{cases} u_t = \Delta(u^m) - k^{-\mu} u^p & \text{in } S, \\ u(x,0) = \phi_k(x) & \text{in } \mathbf{R}^n, \end{cases}$$

where $\mu = (p-m)n - 2 > 0$ and $\phi_k(x) = k^n \phi(kx)$.

Let w_k denote the solution of Problem I_k without the absorption term

$$\begin{cases} u_t = \Delta(u^m) & \text{in } S, \\ u(x,0) = \phi_k(x) & \text{in } \mathbf{R}^n. \end{cases}$$

Then, by the Comparison Principle,

$$u_k \leqq w_k \qquad \text{in } \bar{S} \setminus \{(0,0\}.$$

Since

$$\phi_k(x) \to \|\phi\|_{L^1} \delta(x) \qquad \text{as } k \to \infty$$

in the sense of distributions, it follows from a result due to Friedman and Kamin [10] that

$$w_k \to E_c \qquad \text{as } k \to \infty,$$

where $c = \|\phi\|_{L^1}$, uniformly on compact subsets of $\bar{S} \setminus \{(0,0)\}$. Thus, the sequence $\{u_k\}$ is locally bounded in $\bar{S} \setminus \{(0,0)\}$, and we can extract a subsequence $\{u_{k'}\}$ which converges to a function $U \in C(\bar{S} \setminus \{(0,0)\})$ as $k' \to \infty$ uniformly on compact subsets of $\bar{S} \setminus \{(0,0)\}$.

It is readily verified that U satisfies the Porous Media Equation (1.3) in the sense of distributions in S, and that, because $U \leqq E_c$ in $\bar{S} \setminus \{(0,0)\}$,

$$U(x,0) = 0 \qquad \text{for } x \in \mathbf{R}^n \setminus \{0\}. \tag{5.2}$$

To complete the description of U at $t = 0$, we consider the solutions u_k of Problem I_k for any arbitrary $k > 0$. Proceeding as in [14] we deduce that for any $t > 0$, u_k satisfies

$$\begin{aligned} \int u_k(x,t)dx &= \int \phi_k(x)dx - k^{-\mu} \int_0^t \int u_k^p(x,\tau)dxd\tau \\ &= \int \phi_k(x)dx - \int_0^{k^{\delta n}t} \int u^p(y,s)dyds \end{aligned} \tag{5.3}$$

where the spatial integrals are all taken over $\mathbf{R}^n$. If we let $k \to \infty$ in (5.3) we obtain

$$\int U(x,t)dx = \|\phi\|_{L^1} - \int_0^\infty \int u^p(y,s)dyds.$$

Remembering (5.2) we can conclude that

$$U(x,0) = c_0 \delta(x)$$

where

$$c_0 = \| \phi \|_{L^1} - \int_0^\infty \int u^p(y,s)dyds,$$

and hence that $U = E_{c_0}$.

The proof of Theorem 3 is completed in the usual way (see for instance the proof of Theorem 2) and we omit the details.

6. Generalization

In this section we replace the nonlinear sink term $-u^p$ in Problem I by a more general sink term $-g(u)$. Thus we consider the problem

$$(\mathrm{I}^*)\begin{cases} u_t = \Delta(u^m) - g(u) & \text{in } S \\ u(x,0) = \phi(x) & \text{in } \mathbf{R}^n \end{cases}$$

in which ϕ is a given bounded nonnegative function and g a C^1 function such that

$$g(0) = 0 \quad \text{and} \quad g(u) > 0 \qquad \text{if } u > 0.$$

Depending on the behaviour of $g(u)$ as $u \to 0$, one can derive results, akin to those obtained in the previous sections.

We begin with a result of the kind proved in section 3.

THEOREM 4. *Suppose for some $p > m \geqq 1$,*

$$s^{-p}g(s) \to \sigma \qquad \text{as } s \to 0 \tag{6.1}$$

and suppose ϕ has the property

$$|x|^{2/(p-m)}\phi(x) \to \infty \qquad \text{as } |x| \to \infty.$$

Then the solution u of Problem I^* satisfies

$$t^{1/(p-1)}u(x,t) \to c^*\sigma^{-1/(p-1)} \qquad \text{as } t \to \infty,$$

where $c^* = (1/(p-1))^{1/(p-1)}$, uniformly on sets

$$\{x \in \mathbf{R}^n : |x| \leqq at^{1/\beta}\}, \qquad a \geqq 0, \quad t \geqq 0$$

where $\beta = 2(p-1)/(p-m)$.

PROOF. Let $\| \phi \|_\infty = a$ and let $y(t)$ be the solution of the problem

$$y' = -g(y), \qquad y(0) = a.$$

Then

$$\int_{y(t)}^{a} \frac{ds}{g(s)} = t$$

and by (6.1),

(6.2) $$t^{1/(p-1)}y(t) \to c^*\sigma^{-1/(p-1)} \quad \text{as } t \to \infty.$$

By the maximum principle

$$u(x,t) \leqq y(t),$$

and therefore

(6.3) $$\limsup_{t\to\infty} t^{1/(p-1)}u(x,t) \leqq c^*\sigma^{-1/(p-1)}.$$

Next, choose $d > 0$ so that for $s \in (0, d)$,

(6.4) $$g(s)s^{-p} \leqq \sigma + \varepsilon$$

where ε is some arbitrary small number, and set

$$\phi_d(x) = \min\{\phi(x), d\}.$$

Then, if u_d is the solution of Problem I* with initial value ϕ_d, we have

(6.5) $$u_d(x,t) \leqq d$$

and

(6.6) $$u_d(x,t) \leqq u(x,t).$$

By (6.4) and (6.5),

$$g(u_d)u_d^{-p} \leqq \sigma + \varepsilon$$

whence u_d satisfies the equation

$$u_t = \Delta(u^m) - c(u)u^p$$

where $c(u) \leqq \sigma + \varepsilon$. Therefore, by the maximum principle

$$u_d \geqq w$$

where w satisfies

$$w_t = \Delta(w^m) - (\sigma + \varepsilon)w^p,$$

$$w(x,0) = \phi_d(x) \quad \text{in } \mathbf{R}^n.$$

It follows from the results of section 3 that

$$t^{1/(p-1)}w(x,t)\to c^*(\sigma+\varepsilon)^{-1/(p-1)} \qquad \text{as } t\to\infty,$$

and therefore, that

$$\begin{aligned}\liminf_{t\to\infty} t^{1/(p-1)}u(x,t) &\geqq \liminf_{t\to\infty} t^{1/(p-1)}u_d(x,t)\\ &\geqq c^*(\sigma+\varepsilon)^{-1/(p-1)}.\end{aligned} \tag{6.7}$$

Because ε was an arbitrary positive number (6.7) and (6.3) imply the desired estimate.

The following result generalizes Theorem 2.

THEOREM 5. *Suppose for some* $\alpha < n$

$$s^{-(m+2/\alpha)}g(s)\to 0 \qquad \textit{as } s\to 0,$$

$m>1$ *and* ϕ *satisfies* (H1) *and* (H3). *Then the solution* u *of Problem* I^* *satisfies*

$$t^{\alpha/\gamma}u(x,t)\to f\left(\frac{|x|}{t^{1/\gamma}};A\right) \qquad \textit{as } t\to\infty$$

uniformly on sets of the form $\{x\in\mathbf{R}^n : |x|\leqq at^{1/\gamma}\}$, *where* $\gamma=(m-1)\alpha+2$, *and* f *is the solution of Problem III.*

The proof differs little from that of Theorem 2, hence we omit it.

Finally, Theorem 3 becomes for solutions of Problem I*

THEOREM 6. *Suppose*

$$s^{-(m+2/n)}g(s)\to 0 \qquad \textit{as } s\to 0,$$

$m>1$ *and* $\phi\in L^1(\mathbf{R}^n)$. *Then the solution* u *of Problem* I^* *has the property*

$$t^{1/\delta}\,|u(x,t)-E_{c0}(x,t)|\to 0 \qquad \textit{as } t\to\infty$$

uniformly on sets of the form $\{x\in\mathbf{R}^n : |x|\leqq at^{1/\delta n}\}$, $a\geqq 0$, *where* $\delta=m-1+(2/n)$, E_{c0} *the Barenblatt–Pattle solution with mass* c_0 *and*

$$c_0=\|\phi\|_{L^1}-\int_0^\infty\int_{\mathbf{R}^n} g(u(x,t))\,dxdt.$$

EXAMPLES. Suppose

$$g(u)=u^p+u^q, \qquad q>p>m+2/n, \quad m>1$$

and ϕ satisfies (H1) and (H3). Then, depending on whether $\alpha < 2/(p-m)$, $2/(p-m) < \alpha < n$ or $\alpha > n$, the large time behaviour of the solution u of Problem I* is characterized by, respectively, Theorem 4, 5 or 6.

If

$$g(u) = |\log u|\, u^p, \qquad p > m + 2/n,$$

Theorems 5 and 6 can be used to determine the large time behaviour of u when, respectively, $2/(p-m) < \alpha < n$ or $\alpha > n$.

As in [16] we may also allow the constant A in (H1) to vary with the angle ω of the ray along which $x \to \infty : A = A(\omega)$.

(H1*) For every fixed $\omega \in \mathbf{R}^n$, $|\omega| = 1$

$$\lim_{x\to\infty} |x|^\alpha \phi(|x|\,\omega) = A(\omega)$$

in which $A(\omega) \geqq 0$ ($\not\equiv 0$).

Proceeding as in section 4 we obtain:

THEOREM 7. *Suppose m and p satisfy* (4.1) *and ϕ satisfies* (H1*) *and* (H3) *in which α satisfies* (4.2). *Then*

$$t^{\alpha/\gamma} u(x,t) \to h\left(\frac{x}{t^{1/\gamma}}\right) \qquad \text{as } t \to \infty,$$

uniformly on sets of the form $\{x \in \mathbf{R}^n, |x| \leqq at^{1/\gamma}\}$, $a \geqq 0$. *Here $h(\xi)$ is a positive solution of the problem*

$$\text{(IV)} \begin{cases} \Delta h^m + \dfrac{1}{\gamma}\xi \cdot \nabla h + \dfrac{\alpha}{\gamma} h = 0, & \xi \in \mathbf{R}^n, \\ \lim\limits_{|\xi|\to\infty} |\xi|^\alpha h(|\xi|\, w) = A(\omega). \end{cases}$$

Let us note that the existence and uniqueness of the positive solution of Problem IV is ensured by the proof of Theorem 7. This result is interesting by itself.

Finally, very recently the case $m = 1$ was studied again in [11]. In that paper, also a few observations were made about the case $m > 1$.

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